Titolo della Tesi: inserire il titolo

Sottotitolo

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Hamiltonian mechanics, notation

The Hamilton equations:

$$\begin{pmatrix} q^i_t \\ p_{i,t} \end{pmatrix} = \omega_{ij} \begin{pmatrix} \dfrac{\partial H}{\partial q^j} \\ \dfrac{\partial H}{\partial p_j} \end{pmatrix}$$

The symplectic form $\omega = (\omega_{ij})$: $\omega_{ji} = -\omega_{ij}$ and $d\omega = 0$.

The Poisson tensor $P = (\omega^{ij}) = (\omega_{ij})^{-1}$ has vanishing Schouten bracket: $[P, P] = 0$

The Poisson bracket: $\{f_i, f_j\} = P(df_i, df_j) = \omega^{kh} \dfrac{df_i}{dz^k} \dfrac{df_j}{dz^h}$, $z^h = q^h$ or $z^h = p_h$.

Liouville integrability: n independent conserved quantities f_i in involution, $\{f_i, f_j\} = 0$.

Theorem 1

$$|C(\mathcal{R},\mathcal{R})| = |\mathcal{R}|$$

The set of real continuous functions over the reals has the same cardinality of the continuum.

History of Integrability for PDEs

Integrability for Partial Differential Equations is defined as the existence, for a given equation, of an infinite sequence of symmetries or conserved quantities in involution.

- **goossens93**. (*done with biblatex*)
- CS Gardner, JM Greene, MD Kruskal, RM Miura (1967-11-06) *Method for Solving the Korteweg-deVries Equation*. Physical Review Letters. **19** (1967), 1095–1097.
- P Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Applied Math., **21** (5) (1968), 467–490.
- VE Zakharov, LD Faddeev, *Korteweg–de Vries equation: A completely integrable Hamiltonian system*, Funktsional. Anal. i Prilozhen., 5:4 (1971), 18–27; Funct. Anal. Appl., 5:4 (1971), 280–287.

Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\, dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta \mathcal{H}}{\delta u^j}\right), \quad \text{with} \quad \frac{\delta \mathcal{H}}{\delta u^j} = (-1)^\sigma \partial_\sigma \frac{\partial h}{\partial u^j_\sigma}$$

where $A = (A^{ij})$ is a Hamiltonian operator, i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), with further properties.

Hamiltonian operators

A is a Hamiltonian operator if and only if

$$\{F, G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j}\, dx$$

is a Poisson bracket (skew-symmetric and Jacobi).
$\{,\}_A$ is a Poisson bracket if and only if:

- A is skew-adjoint: $A^* = -A$, where

$$A^*(\psi)^j = (-1)^\sigma \partial_\sigma \left(A^{ij\sigma} \psi_i \right)$$

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- The variational Schouten bracket vanishes:

$$[A,A](\psi^1,\psi^2) = 2\left[\frac{\partial A^{ij\sigma}}{\partial u^l_\tau} \partial_\sigma(\psi^1_j) \partial_\tau (A^{lk\mu} \partial_\mu(\psi^2_k)) \right] = 0$$

(the r.h.s. is defined up to total derivatives $\partial_x(B)$).

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- There is an analogue of KAM theory.

You can use, (without abusing) block of text, with colored background and title.

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Alert Block

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